A THEORY OF THE INITIAL MASS FUNCTION FOR STAR FORMATION IN MOLECULAR CLOUDS

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Abstract

We present a class of models for the initial mass function (IMF) for stars forming within molecular clouds. This class of models uses the idea that stars determine their own masses through the action of powerful stellar outflows. This concept allows us to calculate a semi-empirical mass formula (SEMF), which provides the transformation between initial conditions in molecular clouds and the final masses of forming stars. For a particular SEMF, a given distribution of initial conditions predicts a corresponding IMF. In this paper, we consider several different descriptions for the distribution of initial conditions in star forming molecular clouds. We first consider the limiting case in which only one physical variable – the effective sound speed – determines the initial conditions. In this limit, we use observed scaling laws to determine the distribution of sound speed and the SEMF to convert this distribution into an IMF. We next consider the opposite limit in which many different independent physical variables play a role in determining stellar masses. In this limit, the central limit theorem shows that the IMF approaches a log-normal form. Realistic star forming regions contain an intermediate number of relevant variables; we thus consider intermediate cases between the two limits. Our results show that this picture of star formation and the IMF naturally produces stellar mass distributions that are roughly consistent with observations. This paper thus provides a calculational framework to construct theoretical models of the IMF.

Subject headings: stars: formation – ISM: clouds – galaxies: formation

1. INTRODUCTION

The initial mass function (IMF) is perhaps the most important result of the star formation process. A detailed knowledge of the initial mass function is required to understand galaxy formation, the chemical evolution of galaxies, and the structure of the interstellar medium. Unfortunately, however, the current theory of star formation says very little about the IMF (see, e.g., the reviews of Shu, Adams, & Lizano 1987, hereafter SAL; Zinnecker, McCaughrean, & Wilking 1993). In particular, we remain unable to calculate the initial mass function from first principles.

Given the extreme importance of the IMF and the many successes of the current theory of star formation, we feel that it is now time to begin building models of the IMF. The purpose of this present paper is to present a class of IMF models which use the idea that stars, in part, determine their own masses through the action of powerful stellar winds and outflows (see, e.g., SAL; Lada & Shu 1990). Within the context of the current theory of star formation described below (§1.2), we can conceptually divide the process which determines the IMF into two subprocesses:

[1] The spectrum of initial conditions produced by molecular clouds (the star forming environment).

[2] The transformation between a given set of initial conditions and the properties of the final (formed) star. This transformation is accomplished through the action of stellar winds and outflows.

Notice that molecular clouds are not observed to be collapsing as a whole; on average, the lifetime of a molecular cloud is (at least) an order of magnitude longer than the free-fall time (e.g., Zuckerman & Palmer 1974). Thus, these clouds exhibit quasi-static behavior and it makes sense to conceptually divide the process of determining the distribution of stellar masses into the two steps given above (see Zinnecker 1989, 1990).

A large body of previous work on the IMF exists in the literature (see, e.g., the reviews of Zinnecker, McCaughrean, & Wilking 1993; Elmegreen 1985). Many of these studies use the idea that fragmentation of clouds leads directly to the masses of the forming stars (e.g., Hoyle 1953; Larson 1973; Bodenheimer 1978; Elmegreen & Mathieu 1983). More recent work (Larson 1992, 1995) has extended these ideas to include the observed fractal and hierarchical structure of molecular clouds (e.g., Scalo 1985; Dickman, Horvath, & Margulis 1990; Scalo 1990; Lada, Bally, & Stark 1991; Houlahan & Scalo 1992). Zinnecker (1984, 1985, 1989, 1990) has discussed the two subprocesses given above and has explored several different fragmentation schemes to produce the IMF. The concept that stars help determine their own masses has just now begun to be incorporated into models of the IMF. Silk (1995) has discussed the IMF for stars which have masses limited by feedback due to both ionization and protostellar outflows. Nakano, Hasegawa, & Norman (1995) have introduced a model in which stellar masses are sometimes limited by the mass scales of the formative medium and are sometimes limited by feedback. Finally, a more primitive version of this current theory has been presented previously (Adams 1995).

For the point of view of the IMF adopted in this paper, traditional arguments based

on the Jeans mass are not applicable. A characteristic feature of molecular clouds is that they are highly non-uniform; clumpiness and structure exist on all resolvable spatial scales. In particular, no characteristic density exists for these clouds and hence no (single) Jeans mass exists. We stress that, at least in the context of present day star formation in molecular clouds, *the Jeans mass has virtually nothing to do with the masses of forming stars.*

1.1 The IMF Observed

We begin this discussion by emphasizing that stars can only exist in a finite range of masses. Stellar objects with masses less than about 0.08 $M_\odot$ cannot produce central temperatures hot enough for the fusion of hydrogen to take place; objects with masses less than this hydrogen burning limit are brown dwarfs (see, e.g., Burrows, Hubbard, & Lunine 1989; Burrows et al. 1993; Laughlin & Bodenheimer 1993). On the other end of the possible mass range, stars with masses greater than about 100 $M_\odot$ cannot exist because they are unstable (e.g., Phillips 1994). Thus, stars are confined to the mass range

$$0.08 \le m \le 100\,, \tag{1.1}$$

where we have defined $m \equiv M_*/(1 M_\odot)$. Notice that this mass range is rather narrow in the sense that it is much smaller than the conceivable range of masses. Stars form within galaxies which have masses of about $10^{11} M_\odot$ and stars are made up of hydrogen atoms which have masses of about 10^{-24} g $\sim 10^{-57} M_\odot$. Thus, galaxies *could* build objects anywhere in the mass range from $10^{-57} M_\odot$ to $10^{11} M_\odot$, a factor of 10^{68} in mass scale. And yet, as we have discussed above, stars live in the above mass range which allows stellar masses to vary by only a factor of $\sim 10^3$.

The initial mass function in our galaxy has been estimated empirically. The first such determination (Salpeter 1955) showed that the number of stars with masses in the range m to $m + dm$ is given by the power-law relation

$$f(m)\,dm \sim m^{-b}\,dm\,, \tag{1.2}$$

where the index $b = 2.35$ for stars in the mass range $0.4 \le m \le 10$. However, more recent work (e.g., Miller & Scalo 1979; Scalo 1986; Rana 1991; Tinney 1995) suggests that the mass distribution deviates from a pure power-law. The distribution becomes flatter (and may even turn over) at the lowest stellar masses (b approaches unity for the lowest masses $0.1 \le m \le 0.5$) and becomes steeper at the highest stellar masses ($b \sim 3.3$ for $m > 10$). The observed IMF can be approximated with an analytic fit using a log-normal form (Miller & Scalo 1979), i.e.,

$$\log_{10} f(\log_{10} m) = a_0 - a_1 \log_{10} m - a_2 (\log_{10} m)^2\,, \tag{1.3}$$

where $a_0 = 1.53$, $a_1 = 0.96$, and $a_2 = 0.47$. The true IMF has more structure than a simple log-normal form (Scalo 1986; Rana 1991), although equation [1.3] provides a good analytic reference distribution. Figure 1 shows three successive approximations to the observed IMF: the Salpeter power-law [1.2], the Miller/Scalo log-normal form [1.3],

and the more recent distribution taken from Table 2 of Rana (1991). We note that the construction of the IMF from observational quantities (e.g., the observed luminosity function) requires considerable processing. However, the basic features of the IMF seem to be very robust. As a general rule, the IMF does not change very much from one star forming region to another. For the sake of definiteness, in this paper, we use the analytic fit given by equation [1.3] as a benchmark with which to compare our theoretical models.

1.2 The Current Theory of Star Formation

In the last decade, a generally successful working paradigm of star formation has emerged (see, e.g., SAL for a review). Since the IMF models of this paper use this paradigm as a starting point, in this section we quickly review its basic features. One result of this present work is thus a consistency check – we show that this star formation paradigm *can* produce an IMF similar to that observed.

In our galaxy today, star formation takes place in molecular clouds. These clouds thus provide the initial conditions for the star forming process. Molecular clouds have very complicated substructure. In addition, these clouds exhibit molecular linewidths Δv which contain a substantial non-thermal component (e.g., Myers & Fuller 1992); this linewidth broadening is generally interpreted as a "turbulent" contribution to the velocity field.

Molecular clouds are supported against their self-gravity by both "turbulent" motions and by magnetic fields. The fields gradually diffuse outward (relative to the mass) and small centrally condensed structures known as molecular cloud cores are formed. These cores represent the initial conditions for protostellar collapse. In the simplest picture, these cores can be (roughly) characterized by two physical variables: the effective sound speed $a_{\rm eff}$ and the rotation rate Ω. The effective sound speed generally contains contributions from both magnetic fields and "turbulence", as well as the usual thermal contribution. The total effective sound speed can thus be written

$$a_{\rm eff}^2 = a_{\rm therm}^2 + a_{\rm mag}^2 + a_{\rm turb}^2 \, . \tag{1.4}$$

The molecular cloud cores eventually undergo dynamic collapse, which proceeds from inside-out; in other words, the central parts of the core fall in first and successive outer layers follow as pressure support is lost from below (Shu 1977). Since the infalling material contains angular momentum (the initial state is rotating), not all of the infalling material reaches the stellar surface. The material with higher specific angular momentum collects in a circumstellar disk. The collapse flow is characterized by a well defined mass infall rate $\dot{M}$, the rate at which the central object (the forming star/disk system) gains mass from the infalling core. Notice that no mass scale appears in the problem, only a mass infall *rate*. In particular, the total amount of mass available to a forming star is generally much larger than the final mass of the star.

One important characteristic of the rotating infalling flow described above is that the ram pressure of the infall is weakest at the rotational poles of the object. The central star/disk system gains mass until it is able to generate a powerful stellar wind which

breaks through the infall at the rotational poles and thereby leads to a bipolar outflow configuration. Although the mechanism which generates these winds remains under study (see the review of Königl & Ruden 1993), the characteristics of outflow sources have been well studied observationally (see the review of Lada 1985). One of the basic working hypotheses of star formation theory is that these outflows help separate nearly formed stars from the infalling envelope and thereby determine, in part, the final masses of the stars (SAL; Lada & Shu 1990). In this paper, we use this idea as the basis for calculating a transformation between the initial conditions in a molecular clouds core and the final mass of the star produced by its collapse (see §2; Shu, Lizano, & Adams 1987, hereafter SLA; Adams 1995).

1.3 Organization of the Paper

This paper is organized as follows. In §2, we derive a semi-empirical mass formula which provides a transformation between the initial conditions in a star forming region and the final masses of the stars formed. In subsequent sections, we use this transformation in conjunction with the observed properties of molecular clouds to derive an initial mass function. In §3, we first consider the limit in which only one physical variable (the effective sound speed) determines stellar masses. In §4, we consider the opposite limit in which a large number n of physical variables contribute to the determination of stellar masses; in the limit $n \to \infty$, the central limit theorem implies that the IMF approaches a log-normal form. In §5, we explore more complicated models of the IMF; these models are intermediate between the limiting cases studied in the two previous sections. In this section we also consider the effects of binary companions on the IMF. Finally, we conclude in §6 with a summary and a discussion of our results.

2. A SEMI-EMPIRICAL MASS FORMULA

In this section, we calculate the transformation between initial conditions and the final masses of the stars produced. In other words, we derive a semi-empirical mass formula (SEMF) for the masses of forming stars. In the current picture of star formation, the final masses of stars are produced in part through the action of powerful outflows. Thus, we must know how this stellar outflow stops the inflow and separates the star/disk system from its molecular environment. Although this process has not been well studied, we obtain a working estimate by balancing the ram pressure of the outflow against that of the infall. For this calculation, we adopt the arguments first presented by SLA (see also Adams 1995).

The key concept in this argument is that the final mass of a star is determined by the condition that the stellar outflow is strong enough to reverse the direct infall onto the star. We write this condition in the form

$$\dot{M}_w = \delta \dot{M}_*, \tag{2.1}$$

where $\dot{M}_w$ is the mass loss rate of the wind and $\dot{M}_*$ mass infall rate onto the star itself; this infall rate is generally only a fraction of the total mass infall rate because much of the infalling material falls directly onto the disk. Notice that we should really compare

the ram pressure ($\sim \dot{M}_w v_w$) of the wind with that of the infall ($\sim \dot{M}_* v_*$). However, both velocities are determined by the depth of the stellar potential well and are thus comparable in magnitude; we thus divide out the velocities and incorporate any uncertainties into the parameter δ.

Since we do not yet fully understand how high velocity outflows are produced, we must proceed in a semi-empirical manner (although considerable progress in this area has recently been made – see Shu et al. 1988, 1994). The kinetic energy $E_{\rm out}$ of the outflow will generally be some fraction α of the binding energy of the star, i.e.,

$$E_{\rm out} = \alpha \frac{G M_*^2}{R_*} \, . \tag{2.2}$$

The natural time scale associated with stellar processes is the Kelvin-Helmholtz time scale. We thus take the duration of the outflow (which is produced by a stellar process) to be a fraction β of the Kelvin-Helmholtz time, i.e.,

$$\tau_{\rm out} = \beta \frac{G M_*^2}{R_* L_*} \, . \tag{2.3}$$

Combining the above two equations, we thus reproduce the observational correlation that the mechanical outflow luminosity $L_{\rm out} \equiv E_{\rm out}/\tau_{\rm out}$ is roughly a constant fraction of the photon luminosity L_* of the central source, i.e.,

$$L_{\rm out} = \frac{\alpha}{\beta} \, L_* , \tag{2.4}$$

where observations show that $\alpha/\beta \sim 10^{-2}$ and that this correlation holds over several decades of L_* (see Bally & Lada 1983; Lada 1985; Levreault 1988; Edwards, Ray, & Mundt 1993). Finally, if the winds roughly conserve energy while driving bipolar outflows, we obtain the result

$$\dot{M}_w \frac{G M_*}{R_*} = \epsilon \frac{\alpha}{\beta} L_* , \tag{2.5}$$

where ϵ is an additional efficiency parameter.

The strength of the infall can be measured by the rate $\dot{M}_*$ at which matter falls *directly* onto the star. The total infall rate onto the central star/disk system is given by the collapse solution for an isothermal cloud core (Shu 1977). This infall rate $\dot{M}$ for purely spherical infall takes the form

$$\dot{M} = m_0 a^3 / G \, , \tag{2.6}$$

where $m_0 = 0.975$ is a dimensionless constant. For cloud cores which are not isothermal, the mass infall rate can still be written in the form [2.6] with the sound speed a taken to be the total effective sound speed (see equation [1.4]) and a different numerical constant (Adams et al. 1995).

When rotation is present, not all of the material falls all the way in to the stellar surface (see Cassen & Moosman 1981; Terebey, Shu, & Cassen 1984). The material with

higher specific angular momentum collects in a circumstellar disk whose radius is roughly given by the centrifugal radius

$$R_C \equiv \frac{G^3 M^3 \Omega^2}{16 a^8} \, . \tag{2.7}$$

When the stellar radius is small compared to the centrifugal radius, $R_* \ll R_C$, the direct infall rate $\dot{M}_*$ onto the star itself is a small fraction of the total mass infall rate and is given by

$$\dot{M}_* = \frac{R_*}{2R_C} \dot{M} = \frac{8 m_0 R_* a^{11}}{G^4 M^3 \Omega^2} \, . \tag{2.8}$$

The first equality is taken from equation [24] of Adams & Shu (1986); the second equality follows from the expressions for $\dot{M}$ and R_C.

In general, the stellar mass M_* is only a fraction of the total mass M that has collapsed to the central star/disk system at a given time; we write this condition in the form

$$M_* = \gamma M \, . \tag{2.9}$$

Disk stability considerations (Adams, Ruden, & Shu 1989; Shu et al. 1990) greatly limit the allowed range of the fraction γ. We generally expect $\gamma \sim 2/3$; smaller values of γ imply larger relative disk masses and hence systems that are gravitationally unstable.

The combination of the above results implies that the final properties of the newly formed star are given by the following SEMF:

$$L_* M_*^2 = 8 m_0 \gamma^3 \delta \frac{\beta}{\alpha \epsilon} \frac{a^{11}}{G^3 \Omega^2} = \Lambda \, \frac{a^{11}}{G^3 \Omega^2} \, , \tag{2.10}$$

where we have defined a new dimensionless parameter Λ in the second equality. Under most circumstances, we expect that the parameters α, β, γ, δ, and ϵ can be estimated to a reasonable degree of accuracy. For example, disk stability arguments suggest that $\gamma \sim 2/3$ and empirical estimates imply that $\beta/\alpha \sim 10^2$. The parameter ϵ is close to unity, whereas the parameter δ has a value of a few. We thus expect that the parameter Λ will lie in the range $10^2 \le \Lambda \le 10^3$. In real star forming environments, the parameters α, β, ... do not have exactly the same values for all cases. Instead, the values of these parameters have a *distribution* which is determined by the underlying physics of the problem.

Equation [2.10] provides us with a transformation between initial conditions (the sound speed a and the rotation rate Ω) and the final properties of the star (the luminosity L_* and the mass M_*). If we use "typical" values for present day clouds (e.g., $a = 0.35$ km s^{-1} and $\Omega \sim 3 \times 10^{-14}$ rad s^{-1} ~ 1 km s^{-1} pc^{-1}) and the observed protostellar luminosities ($L_* \sim 20 L_\odot$), we obtain stellar mass estimates $M_* \sim 1 \, M_\odot$, which is the typical mass of stars forming in regions with these properties. In spite of its highly idealized nature, the SEMF [2.10] thus provides reasonable estimates for the mass M_* as a function of initial conditions (a, Ω). It is useful to write this transformation in dimensionless form

$$\widetilde{L} \, m^2 \; = \; 20 \Lambda_3 \, a_{35}^{11} \, \Omega_1^{-2} \, , \tag{2.11}$$

where we have defined $\widetilde{L} \equiv L_*/(1 L_\odot)$, $m \equiv M_*/(1 M_\odot)$, $a_{35} \equiv a/(0.35$ km s$^{-1})$, $\Omega_1 \equiv \Omega/(1$ km s^{-1} pc$^{-1})$, and finally $\Lambda_3 \equiv \Lambda/10^3$.

Notice that much of the uncertainty in this calculation has been encapsulated in the parameter Λ, which should really be considered as a complicated function of all the environmental parameters. Notice also that we have set the final mass of the star by the criterion that its outflow is sufficiently powerful to overwhelm the infall; in actuality, the star will continue to gain some mass, both from residual infall and from disk accretion, after this evolutionary state has been reached. This mass correction can be absorbed into the factor Λ in equation [2.10]. We have also characterized the initial conditions by only two physical variables (a and Ω) whereas much more complicated initial states are possible. Finally, we have ignored radiation pressure in this argument; for sufficiently massive stars ($M_* \geq 7\ M_\odot$) radiation pressure will help the outflow reverse the infall (e.g., see Wolfire & Cassinelli 1986, 1987; Nakano 1989; Jijina & Adams 1995).

In order to evaluate the semi-empirical mass formula derived above, we must determine the relationship between mass and luminosity for young stellar objects. In general, the luminosity has many contributions (Stahler, Shu, & Taam 1980; Adams & Shu 1986; Adams 1990; Palla & Stahler 1990, 1992). For our present purposes, however, we can simplify the picture considerably. The most important source of luminosity for low mass objects is ultimately from infall; in other words, infalling material falls through the gravitational potential well of the star (and disk) and converts energy into photons. This luminosity can be written

$$L_A = \eta \frac{G M \dot{M}}{R_*} \approx 70 L_\odot\ \eta\, a_{35}^2\, m\,, \tag{2.12}$$

where a_{35} and m are the dimensionless sound speed and mass as defined above. For the stellar radius, we have used the scaling relation $R_* = (3 \times 10^{11}\ \text{cm}\ )\, a_{35}$, indicated by the stellar structure calculations of Stahler, Shu, and Taam (1980). The efficiency parameter η is the fraction of the total available energy that is converted into photons. For spherical infall, all of the material reaches the stellar surface and $\eta \approx 1$. For infall which includes rotation, some of the energy is stored in the form of rotational and gravitational potential energy in the circumstellar disk. We generally expect $\eta \sim 1/2$.

In addition, the star can generate its own internal luminosity through deuterium burning, gravitational contraction, and eventually hydrogen burning. This additional luminosity contribution is important for stars with masses larger than $\sim$ few $M_\odot$. For low mass stars on the main sequence, the luminosity is a very sensitive function of stellar mass, $L_* \sim M_*^4$. For higher mass stars, the mass/luminosity relation flattens to the form $L_* \sim M_*^2$ (and eventually flattens further to the form $L_* \sim M_*$ for very high mass stars). For stars still gaining mass from infall, the internal luminosity contribution is somewhat different, but has been calculated for much of the relevant range of parameter space (Stahler 1983, 1988; Palla & Stahler 1990, 1992; see also Fletcher & Stahler 1994). For our present purposes, we use the following simple approximation for the internal luminosity

$$L_{int} = 1 L_\odot \Big(\frac{M_*}{1 M_\odot} \Big)^4\,, \tag{2.13}$$

which is roughly valid for the mass range $1 M_\odot \leq M_* \leq 10 M_\odot$.

Putting both contributions together, we obtain the luminosity as a function of mass in dimensionless form:

$$\widetilde{L} = 70\,\eta\, a_{35}^2\, m\, + m^4\,. \tag{2.14}$$

Thus, at low masses, the luminosity is dominated by the contribution from infall and $\widetilde{L}$ is a linear function of m. At higher masses $\widetilde{L} \sim m^4$ with the crossover point at $m \approx 3.3$ (for representative values of $\eta = 1/2$ and $a_{35} = 1$). At sufficiently high masses, the form [2.14] is no longer valid and $\widetilde{L} \sim 100m^2$ for the range $10 \le m \le 100$.

Finally, putting all of the above results together, we present the SEMF in dimensionless form:

$$m = 0.66[\Lambda_3/\eta]^{1/3}\, a_{35}^3 \Omega_1^{-2/3} \qquad \text{low }\; m\,, \tag{2.15a}$$

$$m = 1.65\,\Lambda_3^{1/6}\, a_{35}^{11/6}\, \Omega_1^{-1/3} \qquad \text{intermediate }\; m\,, \tag{2.15b}$$

$$m = 0.67\,\Lambda_3^{1/4}\, a_{35}^{11/4}\, \Omega_1^{-1/2} \qquad \text{high }\; m\,. \tag{2.15c}$$

This SEMF provides a transformation between initial conditions and the final mass of the star. One way to view this result is shown in Figure 2. Here, we assume that the effective sound speed a_{35} and the rotation rate Ω_1 are the two most important parameters which determine the initial conditions. We thus set all of the remaining parameters to constant values such that $\Lambda_3 = 1$ and we take $\eta = 1/2$. Figure 2 shows the resulting contours of constant mass in the plane of initial conditions, i.e., the (a_{35}, Ω_1) plane. The region in the far upper left corner of the diagram corresponds to brown dwarfs, i.e., objects with masses less than the hydrogen burning limit. The region in the lower right corner corresponds to stars that are too massive to be stable; the lower right part of the diagram also corresponds to initial conditions for which radiation pressure helps limit the stellar mass (see Jijina & Adams 1995). In actual star forming regions, the parameters (in addition to a_{35} and Ω_1) which enter into the SEMF will have a distribution of values (roughly centered on the values assumed here). As a result, the set of initial conditions which lead to a star of a given mass will be a band in the (a_{35}, Ω_1) plane instead of a line. Finally, for comparison, we note that previous authors have considered the mean density n and the mass infall rate $\dot{M}$ as the two most important variables which determine stellar masses (see Figures 1 – 3 of Nakano et al. 1995).

3. EMPIRICAL MODEL: THE INITIAL MASS FUNCTION FOR CLUMPY MOLECULAR CLOUDS

In this section, we consider the limiting case in which the effective sound speed is the only physical variable which determines stellar masses. Here, we use two observed scaling laws to determine the distribution of the effective sound speed and hence the distribution of initial conditions for star formation. This result, in conjunction with the SEMF derived in the previous section, produces a nearly power-law IMF in reasonable

agreement with observations. We also calculate the efficiency of star formation from this model.

3.1 Observed Distributions of Initial Conditions

Observational work spanning many different star forming regions and many different size scales suggests that the effective sound speed in molecular cloud cores (or clumps) obeys a simple scaling law (see, e.g., Larson 1981; Scalo 1987; Myers & Fuller 1992). For sufficiently large size scales ($r \sim 1$ pc) and low density $n < 10^4$ cm^{-3}, the observed linewidths Δv in cores have a substantial nonthermal component which scales with density according to the law

$$\Delta v \propto \rho^{-1/2}. \tag{3.1}$$

Although this result was obtained from observational data, relations of this type can be calculated theoretically from the supposition that magnetohydrodynamic waves (e.g., Alfvén and magnetoacoustic waves) are the source of the non-thermal motions (see Fatuzzo & Adams 1993; McKee & Zweibel 1995). Notice that this scaling law is valid over a finite range of densities; at sufficiently large densities the observed linewidths become equal to the thermal linewidths (in other words, the total linewidth Δv does not vanish as $\rho \to \infty$ as implied by equation [3.1]).

If this velocity Δv is interpreted as a transport speed, then a "turbulent" or "non-thermal" component to the pressure can be derived (Lizano & Shu 1989; Myers & Fuller 1992) and has the form $P = P_0 \ln(\rho/\rho_0)$. This "turbulent" equation of state also implies a scaling relation between the line-width Δv and the mass $M_{\rm cl}$ of the clump. Using hydrostatic equilibrium arguments, we obtain the relation

$$M_{\rm cl} \sim (\Delta v)^q \, , \tag{3.2a}$$

where $q \approx 4$ for the law given by equation [3.1]. Since we consider the linewidth Δv to define the effective sound speed of the region, we can write this scaling relation in the form

$$M_{\rm cl} = \widehat{M} a_{35}^q \, , \tag{3.2b}$$

where the mass scale $\widehat{M} \sim 7\ M_\odot$ is determined by the normalization of the observed scaling law (see, e.g., Larson 1981; Myers & Fuller 1992).

Given the above result, we must now determine the distribution of clump masses $M_{\rm cl}$. Many groups have studied the observed clump mass spectrum of molecular clouds and have found nearly power-law forms, i.e.,

$$\frac{dN_{\rm cl}}{dM_{\rm cl}} \sim M_{\rm cl}^{-p} \, , \tag{3.3}$$

where the index of the power-law typically has the value $p \approx 3/2$ (e.g., see Scalo 1985; Lada, Bally, & Stark 1991; Blitz 1993; Tatematsu et al. 1993). This scaling relation must have a cutoff at both high mass (to keep the total mass of the cloud finite) and at low mass (to keep the total number of clumps finite).

Although the distribution [3.3] was obtained from observations, such distributions can, in principle, be calculated theoretically. For example, simple models which envisage clouds to be composed of a collection of interacting clumps (Norman et al. 1995) can be used to derive clump mass distributions of the general form [3.3]. We emphasize that much more theoretical work on this subject should be done.

3.2 A Simple Model for the IMF

We can now piece together all of the above arguments to construct an initial mass function. Here, we interpret the line-width Δv as the effective transport speed a which determines the initial condition for star formation. We use the semi-empirical mass formula of §2 to provide the transformation between the initial conditions and the final stellar properties. To start, we consider the simplest case in which only the sound speed varies and the remaining parameters of the SEMF are kept constant. The relationship [3.2] between clump mass and linewidth, in conjunction with the clump mass spectrum of equation [3.3], determines the distribution of the effective sound speed. Combining this distribution with the SEMF, we obtain an initial mass function of the form

$$f = \frac{dN}{dM_*} = \frac{dN}{dM_{\rm cl}} \frac{dM_{\rm cl}}{dM_*} \sim M_*^{-b} \, , \tag{3.4}$$

where we have assumed that only a single star forms in a given clump. The power-law index b of the distribution is given by

$$b = q(p-1)/\mu + 1 \approx 2/\mu + 1 \, , \tag{3.5}$$

where μ is the scaling exponent which determines how the stellar mass varies with effective sound speed. As shown by equation [2.15], this index lies in the range $11/6 \le \mu \le 11/3$. Thus, this simple argument produces a power-law IMF with an index in the range $b = 1.6 - 2.1$. This result compares reasonably well with the observed power-law index of the IMF which has $b \approx 2.35$ (see Salpeter 1955).

In Figure 3, we show the IMF calculated from this model. Here we use equations [3.2] and [3.3] to determine the distribution of the effective sound speed. We then use the SEMF in the form of equation [2.11] and the mass/luminosity relationship [2.14]. The result is shown as the dashed curve in Figure 3; as indicated by equation [3.4], this distribution has a power-law index $b \approx 1.6$ at low masses and $b \approx 2.1$ at higher masses. Also shown for comparison is the analytic fit to the observed IMF (from Miller & Scalo 1979). Notice that the agreement between the theory and the observations is quite reasonable, but is not exact. We interpret this finding to mean that this picture of the IMF is basically correct, but it still incomplete.

The basic logic of this model can be summarized as follows. Molecular clouds produce a distribution of initial conditions for star formation. In the simplest picture considered here, the clouds produce a distribution of clump masses. Because larger clumps have larger effective sound speeds due to turbulence and other small-scale physical processes, this distribution of clump masses implies a corresponding distribution of effective sound

speeds, which represent the initial conditions for star formation. We then use the idea that outflows help determine the final masses of forming stars to find a transformation between the initial conditions and the final stellar masses. Using both this transformation and the set of initial conditions, we thereby obtain the IMF.

3.3 Efficiency of Star Formation

We can directly calculate the efficiency of star formation from this model of the IMF. Here, we define the star formation efficiency $\mathcal{E}$ to be the ratio of the mass in stars to the total cloud mass, i.e.,

$$\mathcal{E} = \int_{M_1}^{M_2} \frac{dN}{dM_{\rm cl}} M_* dM_{\rm cl} \bigg/ \int_{M_1}^{M_2} \frac{dN}{dM_{\rm cl}} M_{\rm cl} dM_{\rm cl} \, , \tag{3.6}$$

where M_1 and M_2 are the lower and upper cutoffs of the clump mass distribution [3.3]. Using the SEMF of the previous section (for simplicity, we use only the low mass version [2.15a]) in conjunction with the scaling law of equation [3.2], we can evaluate this integral to obtain the efficiency

$$\mathcal{E} = 0.66 \, [\Lambda_3/\eta]^{1/3} \, \frac{2-p}{1+3/q-p} \; \widehat{m}^{-3/q} \; {m_2}^{3/q-1} \, , \tag{3.7}$$

where we have assumed that $M_1 \ll M_2$. We have also defined $\widehat{m} = \widehat{M}/(1 M_\odot)$ and $m_2 = M_2/(1 M_\odot)$. If we use representative values, $\Lambda_3 = 1$, $\eta = 1/2$, $p = 3/2$, $q = 4$, $\widehat{m} = 7$, and $m_2 = 1000$ (e.g., see Williams, de Geus, & Blitz 1994), we obtain a star formation efficiency $\mathcal{E} \approx 0.07$.

The star formation efficiency calculated here must be compared with the observed value for giant molecular clouds taken as a whole. This efficiency typically has a value of a few percent (e.g., Duerr, Imhoff, & Lada 1982; Lada, Strom, & Myers 1993). We conclude that this semi-empirical model produces a star formation efficiency in reasonable agreement with observations.

For completeness, we calculate the efficiency of star formation for an individual clump. This efficiency $\mathcal{E}_*$ in the low mass regime is then given by

$$\mathcal{E}_* = \frac{M_*}{M_{\rm cl}} = \frac{0.66}{\widehat{m}} [\Lambda_3/\eta]^{1/3} \, a_{35}^{3-q} \approx 0.12 \, a_{35}^{-1} \, , \tag{3.8}$$

where the final approximate equality was obtained using the representative values $\Lambda_3 = 1$, $\eta = 1/2$, $q = 4$, and $\widehat{m} = 7$. A similar formula can be derived for the regimes of intermediate and high mass stars.

In this model, we have assumed that only a single star forms within a given molecular clump. In many cases (see Zinnecker et al. 1993; Lada et al. 1993; Hillenbrand et al. 1993), a cluster of stars forms within a single clump and the star formation efficiency of the individual clump can be much higher, $\mathcal{E}_{\rm clust} \sim 0.2 - 0.5$. This present model does not take cluster formation into account, although this issue is important and should be addressed in future studies.

4. RANDOM MODEL: THE INITIAL MASS FUNCTION AS A RESULT OF THE CENTRAL LIMIT THEOREM

In this section, we consider the limit in which a large number of physical variables is required to determine stellar masses. We thus adopt a statistical approach to the calculation of the IMF. We start with the semi-empirical mass formula of §2 and consider it to be a product of random variables. In the limit that the number n of random variables is large ($n \to \infty$) and the variables are completely independent, the IMF approaches a log-normal distribution. This result is a direct consequence of the central limit theorem (see, e.g., Richtmyer 1978; Parzen 1960). For the more realistic case of a finite number of not-completely-independent variables, we must study how the resulting distribution differs from a log-normal distribution (see also §5). The idea of using the central limit theorem to obtain a log-normal distribution has been discussed previously in models where the stellar masses are determined by fragmentation (Larson 1973; Elmegreen & Mathieu 1983; Zinnecker 1984, 1985). In this paper, we adopt a different approach using the SEMF of §2 as the starting point for our calculation. We also note that the number n of physical variables is actually finite and hence departures of the IMF from a log-normal form are important.

The semi-empirical mass formula can be written in the general form of a product of variables

$$M_* = \prod_{j=1}^{n} \alpha_j \,, \tag{4.1}$$

where the α_j represent the various quantities on the right hand side of equation [2.10], i.e., α, β, γ, δ, ϵ, a, and Ω (taken to the appropriate powers). In this present discussion, we regard these quantities α_j as a collection of n random variables. Thus, by taking the logarithm of this equation, we find that the logarithm of the mass is a sum of random variables,

$$\ln M_* = \sum_{j=1}^{n} \ln \alpha_j + \mathit{constant} \,, \tag{4.2}$$

where the constant term includes all the quantities in the SEMF that are truly constant (e.g., the gravitational constant G).

The stellar mass is thus determined by a composite random variable that is given by the sum of random variables. The central limit theorem shows that the distribution for the composite variable always approaches a normal (gaussian) distribution as the number n of variables approaches infinity. In order to use this result, we must redefine the basic variables $\ln \alpha_j$ so that the new variables ξ_j have zero mean, i.e.,

$$\int_{-\infty}^{\infty} \xi_j f_j(\xi_j) d\xi_j = 0 \,, \tag{4.3}$$

where f_j is the probability density of the jth variable. The distribution f_j is, in general, not a normal (gaussian) distribution. These new variables ξ_j are related to the old variables α_j through the relation

$$\xi_j \equiv \ln \alpha_j - \langle \ln \alpha_j \rangle \equiv \ln[\alpha_j / \bar{\alpha}_j] \,, \tag{4.4}$$

where angular brackets represent averages. Keep in mind that the averages are taken over the logarithms of the α_j and not over the variables α_j themselves, i.e.,

$$\ln \bar{\alpha}_j = \langle \ln \alpha_j \rangle = \int_{-\infty}^{\infty} \ln \alpha_j \, f_j(\ln \alpha_j) \, d \ln \alpha_j \, . \tag{4.5}$$

Similarly, the distributions f_j are the distributions of $\ln \alpha_j$ and not the distributions of α_j. Each of the rescaled variables ξ_j has a variance σ_j given by

$$\sigma_j^2 \; = \; \int_{-\infty}^{\infty} \xi_j^2 f_j(\xi_j) d\xi_j \, . \tag{4.6}$$

Next we construct a composite random variable ζ, defined by

$$\zeta \equiv \sum_{j=1}^{n} \xi_j \; = \sum_{j=1}^{n} \ln[\alpha_j / \bar{\alpha}_j] \, . \tag{4.7}$$

In terms of this new variable, the semi-empirical mass formula becomes

$$M_* = M_C \; \mathrm{e}^{\zeta} \, , \tag{4.8}$$

where M_C is a characteristic mass scale. The distribution of stellar masses (the IMF) is thus determined by the distribution of the composite variable ζ. The mass scale M_C is determined by the mean values of the logarithms of the original variables α_j, i.e.,

$$M_C \equiv \prod_{j=1}^{n} \exp[\langle \ln \alpha_j \rangle] \equiv \prod_{j=1}^{n} \bar{\alpha}_j \, , \tag{4.9}$$

where we have defined $\bar{\alpha}_j = \exp[\langle \ln \alpha_j \rangle]$.

As we find below, the variance $\langle \sigma \rangle$ of the composite variable ζ essentially determines the width of the stellar mass distribution and is thus of fundamental importance for this present discussion. The variance is given by

$$\langle \sigma \rangle^2 = \sum_{j=1}^{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi_j d\xi_k \, \xi_j \, \xi_k \, f(\xi_j, \xi_k) \, , \tag{4.10}$$

where $f(\xi_j, \xi_k)$ is the joint probability distribution of the variables ξ_j and ξ_k. *If the variables are statistically independent*, then the joint probability is the product of the individual probability distributions and the integral in equation [4.10] can be separated. Using the fact that each of the variables ξ_j has zero mean (equation [4.3]) and the definition [4.6], we thus obtain the total variance,

$$\langle \sigma \rangle^2 = \sum_{j=1}^{n} \sigma_j^2 \, . \tag{4.11}$$

We can now define a new random variable

$$\tilde{\zeta} \equiv \frac{\zeta}{\langle\sigma\rangle} , \tag{4.12}$$

which has zero mean and unit variance. The central limit theorem tell us that the distribution of the composite random variable $\tilde{\zeta}$ approaches a normal distribution in the limit $n \to \infty$, i.e.,

$$f(\tilde{\zeta}) \to \mathcal{N} \mathrm{e}^{-\tilde{\zeta}^2/2} , \tag{4.13}$$

where $\mathcal{N} = 1/\sqrt{2\pi}$ is a normalization constant. This result is *independent* of the initial distributions f_j.

The mass of the star (from equation [4.1]) is related to this new variable $\tilde{\zeta}$ through the relation

$$\ln M_* = \ln M_C + \langle\sigma\rangle\tilde{\zeta} , \tag{4.14}$$

where the variable $\tilde{\zeta}$ now has a known (gaussian) distribution. Combining equations [4.13] and [4.14], we can write the distribution f of stellar masses in the form

$$\ln f(\ln m) = A - \frac{1}{2\langle\sigma\rangle^2} \Big\{ \ln\big[m/m_C\big] \Big\}^2 , \tag{4.15}$$

where A is a constant and where we have defined $m \equiv M_*/(1M_\odot)$ and $m_C \equiv M_C/(1M_\odot)$. Notice that the constant A just sets the overall normalization of the distribution. The shape of the distribution is thus completely determined by the mass scale m_C and the total variance $\langle\sigma\rangle$.

In this limit, the distribution of stellar masses has exactly the same form as the Miller/Scalo approximation of equation [1.3]. The three constants a_1, a_2, and a_3 of the Miller/Scalo law are related to the physical variables A, m_C, and $\langle\sigma\rangle$ through the relations

$$A = a_0 \ln 10 + \frac{a_1^2 \ln 10}{4a_2} \approx 4.65 , \tag{4.16a}$$

$$\langle\sigma\rangle^2 = \frac{\ln 10}{2a_2} \approx 2.45 , \tag{4.16b}$$

$$\ln m_C = -\frac{a_1 \ln 10}{2a_2} \approx -2.35 . \tag{4.16c}$$

Thus, the "characteristic mass scale" of the distribution is $m_C \approx 0.095$.

In this limit, where the SEMF involves a large number of statistically independent variables, we obtain a "pure" log-normal distribution. Since the observed IMF can be approximately fit by a log-normal distribution (see Figure 1), this model of the IMF is in reasonable agreement with observations. In this limit, the only relevant parameters are the total width of the distribution (determined by $\langle\sigma\rangle$) and the center of the distribution

(determined by m_C). The product of the mean values of all the of relevant variables combine to determine the mass scale m_C of the distribution (see equation [4.9]). Similarly, the widths of all of the original variables combine to determine the total width $\langle\sigma\rangle$ of the final distribution (see equation [4.11]).

We thus obtain an important consistency check on this model of the IMF: The total width $\langle\sigma\rangle$ and the characteristic mass scale m_C can be calculated and compared with the values required to fit the observed IMF (see equation [4.16]). The quantities $\langle\sigma\rangle$ and m_C are determined by the distributions of all of the physical variables in the problem. In a complete theory, we could calculate these initial distributions from *a priori* considerations. In the absence of a complete theory, however, we can use observations of the physical variables to estimate their distributions and hence calculate $\langle\sigma\rangle$ and m_C. Such a calculation is performed in Appendix B. Using estimates of the distributions of the observed physical variables $a_{\rm eff}$, Ω, etc., we obtain values $\langle\sigma\rangle \approx 1.8$ and $m_C \approx 0.25$. Although these values are somewhat higher than the values required to fit the Miller/Scalo IMF (see equation [4.16]), we consider the agreement to be quite good, given the crudeness of the calculation.

5. INTERMEDIATE EXAMPLES AND APPLICATIONS

The previous discussion has considered the two limiting cases in which the number n of physical variables that determine stellar masses is either one (§3) or infinite (§4). In the first case, we obtain a nearly power-law IMF; in the second case, we obtain a log-normal IMF. In realistic star forming regions, however, we must consider the intermediate cases with $1 < n < \infty$. In particular, we must determine the form of the composite distribution (the IMF) for these intermediate cases. In general, the answer depends on the distributions of the initial variables. In this section, we explicitly calculate theoretical IMFs for several different cases with various distributions of initial conditions. We also consider the problem of binary companions. Although the theory discussed thus far only applies directly to the formation of single stars, we can show (§5.3) that the inclusion of binary companions does not greatly change the resulting IMF.

5.1 Uniform Distributions

In this subsection, we consider the simple case of n fundamental variables α_j, each with the same distribution $f_j(\ln\alpha_j)$. We also consider the simplest type of distribution in which $\xi_j = \ln\alpha_j$ is uniformly distributed in an interval $[-w, w]$, i.e.,

$$\begin{aligned} f_j(\xi_j) &= \frac{1}{2w} \qquad \text{for } -w \le \xi_j \le w\,, \\ &= 0 \qquad\quad \text{otherwise}\,. \end{aligned} \tag{5.1}$$

The variance σ_j of each individual variable is related to the width w of the interval by the expression

$$\sigma_j^2 = w^2/3\,. \tag{5.2}$$

For a given number n of variables, we can thus obtain the required total width $\langle\sigma\rangle$ of the distribution by taking

$$w^2 = \frac{3}{n}\langle\sigma\rangle^2 \,, \tag{5.3}$$

where we use equation [4.16b] to set the value of $\langle\sigma\rangle$. We also use equation [4.16c] to set m_C and hence the center of the distribution.

The resulting distribution (IMF) is shown in Figure 4 for the case $n = 10$. To obtain this result, we have used a random number generator to produce the SEMF variables (distributed according to equation [5.1]) and have calculated a million (10^6) realizations of the mass. Notice that these random variables *do not* have a gaussian distribution, but the sum of random variables comes rather close to a log-normal distribution and hence reproduces the Miller/Scalo IMF quite well. In other words, $n = 10$ is "close enough to ∞" for the central limit theorem to apply and hence for the distribution of ζ to be nearly gaussian, i.e., for the IMF to have nearly a log-normal form.

Next, we would like to determine how the composite distribution changes with the number n of fundamental variables ξ_j. Although the result depends on the initial distributions f_j of the variables, we can get some feeling for this problem by using the uniform distribution [5.1] and varying the number n. The result is shown in Figure 5 for the cases n=1, 2, 3, and 5. The mass scale m_C has been set to correspond to that of the Miller/Scalo IMF (shown as the solid curve). Notice that the composite distribution converges toward the log-normal limit rather rapidly with increasing values of n. As expected, the largest departures are for high masses, i.e., for the tail of the distribution.

5.2 Power-law Distributions

In this subsection, we explicitly consider the case in which all of the physical variables appearing in the SEMF have power-law distributions. This case is expected to be a reasonable approximation for many astrophysical systems.

We consider each fundamental variable to have a distribution of the form

$$\frac{dN}{d\alpha_j} = C\alpha_j^{-p} \,, \tag{5.4}$$

where C is the normalization constant. If the power-law index $p > 1$, as we assume here, then we must specify the lowest value of the parameter α_{0j}, i.e., we introduce a lower cutoff but not an upper cutoff. Next, we write the distribution in terms of the logarithmic variable ξ_j, which is defined by

$$\xi_j \equiv (p-1)\ln(\alpha_j/\alpha_{0j}) - 1 \,. \tag{5.5}$$

The corresponding distribution f_j is then given by

$$f_j = \mathrm{e}^{-(\xi_j+1)} \,, \tag{5.6}$$

i.e., we obtain an exponential distribution. Notice that this new variable ξ_j has zero mean and unit variance.

For this case, we can explicitly calculate the composite distribution, and hence the IMF, from the initial distributions. Here we employ standard methods from probability theory (e.g., Richtmyer 1978; Parzen 1960). The Fourier transform χ_j of the distribution f_j (χ_j is generally known as the characteristic function of the variable ξ_j) is defined by the usual integral

$$\chi_j(\lambda) \equiv \int_{-1}^{\infty} d\xi \, \mathrm{e}^{-i\lambda\xi} \mathrm{e}^{-(\xi+1)} = \frac{\mathrm{e}^{i\lambda}}{1+i\lambda} \, , \tag{5.7}$$

where we have evaluated the integral to obtain the second equality.

Next, we define a composite variable $\tilde{\zeta}$ according to

$$\tilde{\zeta} \equiv \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \xi_j \ , \tag{5.8}$$

which also has zero mean and unit variance. The Fourier transform χ_ζ of the composite distribution is then the product of the Fourier transforms of the individual distributions evaluated at $\lambda/\sqrt{n}$. Thus, the composite transform χ_ζ is given by

$$\chi_\zeta(\lambda) = \Big[\chi_j(\lambda/\sqrt{n})\Big]^n = \Big[\frac{\mathrm{e}^{i\lambda/\sqrt{n}}}{1+i\lambda/\sqrt{n}}\Big]^n \, . \tag{5.9}$$

To obtain the composite distribution itself $f_n(\tilde{\zeta})$, we simply take the inverse Fourier transform of equation [5.9]. We thus obtain

$$f_n(\tilde{\zeta}) = \frac{\sqrt{n}}{(n-1)!} \Big\{ n + \sqrt{n}\tilde{\zeta} \Big\}^{n-1} \mathrm{e}^{-(n+\sqrt{n}\tilde{\zeta})} \, , \tag{5.10}$$

i.e., we obtain a *gamma distribution* for the composite variable $\tilde{\zeta}$. It is straightforward to show that in the limit $n \to \infty$, the distribution [5.10] approaches a log-normal form (as the central limit theorem requires). However, for the special case of power-law initial distributions, we have an exact form for the distribution for intermediate cases (i.e., for finite values of n).

The total variance $\langle\sigma\rangle$ of the composite distribution depends on the power-law indices p and the number of variables through the relation

$$\langle\sigma\rangle^2 = \frac{n}{(p-1)^2} \, . \tag{5.11}$$

Similarly, the central mass scale m_C of the distribution is given by

$$m_C = \frac{n}{p-1} + \sum_{j=1}^{n} \ln \alpha_{0j} \, . \tag{5.12}$$

In Figure 6, we show the composite distributions f_n for various values of n. Also shown for comparison is the log-normal form which corresponds to the limit $n \to \infty$. All

distributions shown here have the same total variance $\langle\sigma\rangle$ and characteristic mass scale m_C as given by equation [4.16]. Notice that the convergence to the log-normal form is much slower than for the case of uniform distributions considered in the previous section. In particular, for sufficiently large masses, the IMF falls off like a decaying exponential in the variable $\ln m$ (a power-law in the variable m) instead of like a gaussian. We note that the observed IMF seems to have a power-law tail at high masses, although the exact slope is somewhat uncertain (see, e.g., Massey et al. 1995 for a good discussion of this issue).

5.3 The Effects of Binary Companions

In this section, we address the issue of binary companions. Thus far, the discussion of this paper has focused on the formation of a single star. On the other hand, most stars live within binary systems (see, e.g., Abt & Levy 1976; Abt 1983; Duquennoy & Mayor 1991; Bodenheimer, Ruzmaikina, & Mathieu 1993). Several different mechanisms for producing binary companions have been proposed, including capture via star/disk interactions (Clarke & Pringle 1991), formation from gravitational instabilities in circumstellar disks (Adams, Ruden, & Shu 1989), and fragmentation during protostellar collapse (Boss 1992; Bonnell & Bastien 1992). Unfortunately, however, a complete theory of binary formation has not yet been obtained. As a result, we must once again proceed in a semi-empirical manner.

The SEMF of §2 can be interpreted as providing the final mass M_{P*} of the *primary* star in a binary system. If we can write this SEMF in the form of equation [4.1], then the mass of the secondary M_{S*} can also be written in the general form of a product of variables, i.e.,

$$M_{S*} = \alpha_S \prod_{j=1}^{n} \alpha_j \, , \tag{5.13}$$

where $\alpha_S = M_{S*}/M_{P*}$ is the mass ratio of the binary system. Note that $\alpha_S \le 1$ by definition. A complete theory of binary formation would give us a theoretical estimate of the distribution of the mass ratio α_S. As mentioned above, however, we do not yet have a complete theory of binary formation. As a result, we must use observations to measure and/or constrain the distribution of the mass ratio. For a given distribution of α_S, we can determine the consequences for the IMF using the framework developed in this paper.

We first note that if the distribution of masses for the primary star has a known form (e.g., the log-normal form of §4), then the distribution of the secondary masses has nearly the same form. This result follows directly from equation [5.13] for the SEMF and holds for any α_S distribution that is not overly pathological (this statement can be made mathematically more precise – see Richtmyer 1978; Parzen 1960). Furthermore, the width $\langle\sigma\rangle_S$ of this secondary distribution is directly determined from the width $\langle\sigma\rangle_P$ of the primary distribution through the relation

$$\langle\sigma\rangle_S^2 = \langle\sigma\rangle_P^2 + \sigma_{\alpha S}^2 \, , \tag{5.14}$$

where $\sigma_{\alpha S}$ is the variance of the mass ratio distribution. In obtaining this result, we have assumed that the distribution of the mass ratio α_S is independent of the other variables in the problem. If the ratio α_S is not completely independent, then the expression [5.14] provides an upper limit to the variance of the secondary distribution. Notice that the width of the secondary distribution is always wider than that of the primary distribution. Similarly, the characteristic mass scale m_{CS} of the secondary distribution is given by

$$m_{CS}/m_{CP} = \mathrm{e}^{\langle \ln \alpha_S \rangle} \, , \tag{5.15}$$

where m_{CP} is the mass scale of the primary distribution and where angular brackets denote averages over the distribution.

As a reference point, we consider the simplest case in which the mass ratio α_S is uniformly (randomly) distributed over the interval $[0, 1]$. For this case, $\sigma_{\alpha S} = 1$ and $\langle \ln \alpha_S \rangle = -1$. Although the observed distribution of mass ratios is not completely uniform, these values for $\sigma_{\alpha S}$ and $\langle \ln \alpha_S \rangle$ represent reasonable estimates (see, e.g., Duquennoy & Mayor 1991 for a more detailed discussion).

The total mass distribution, including both primary and secondary stars, is the sum of the two individual distributions. For the sake of definiteness, we use the large n limit for which both distributions obtain a nearly log-normal form. In this case, we obtain the total distribution in the form

$$f = \mathcal{N} \Big\{ \mathrm{e}^{-\zeta^2/2} + \mathcal{F} \mathrm{e}^{-(\zeta + \zeta_0)^2/2B^2} \Big\} \, , \tag{5.16}$$

where $\mathcal{N}$ is the normalization constant, $\mathcal{F}$ is the binary fraction, and $\zeta = \ln m/m_{CP}$ is the composite variable for the primary mass distribution (see equations [4.7] and [4.8]). We have also defined a parameter B which represents the ratio of the widths of the secondary and primary mass distributions,

$$B \equiv \left[1 + \sigma_{\alpha S}^2 / \langle \sigma \rangle_P^2 \right]^{1/2} \, , \tag{5.17}$$

and a parameter ζ_0 which determines the difference in the centers of the two distributions,

$$\zeta_0 \equiv - \frac{\langle \ln \alpha_S \rangle}{\langle \sigma \rangle_P} \, . \tag{5.18}$$

In the limit $B \to 1$ and $\zeta_0 \to 0$, we recover the primary distribution for the IMF. For relatively small departures from this limit, the joint distribution f is nearly the same as the primary distribution. For the reference case of a uniform distribution of the mass ratio α_S, we obtain values $B \approx 1.2$ and $\zeta_0 \approx 0.64$.

We can quantify the difference between the joint distribution f and the original distribution f_0 for the masses of the primary stars. We define an error functional E_R,

$$E_R[f] \equiv 2\sqrt{\pi} \int_{-\infty}^{\infty} \big| f_0 - f \big|^2 \, d\zeta \, , \tag{5.19}$$

where we have normalized the integral such that $E_R[f=0]=1$. The size of the error estimate E_R is thus a measure of how the distribution f differs from the original distribution f_0. For the joint distribution given by equation [5.16], we evaluate this functional to obtain

$$E_R = \frac{B^2 \mathcal{F}^2}{(1+B\mathcal{F})^2} \Big\{ 1 + B^{-1} - 2\sqrt{2}(1+B^2)^{-1/2}\, \mathrm{e}^{-\zeta_0^2/2(1+B^2)} \Big\} . \tag{5.20}$$

Using the uniform distribution to estimate ζ_0 and B, we find $E_R \approx 0.04$. We thus conclude that the effects of binary companions on the IMF are not overly large for the paradigm considered in this paper.

In some sense, the error estimate obtained above is overly conservative because it includes the differences between the two distributions over the entire mass range $0 \le m \le \infty$. If we normalize the two distributions to unity for a mass of 1.0 $M_\odot$ and only consider the expected mass range for stars, the difference is much smaller. This result is shown in Figure 7. The solid curve shows the primary distribution f_0 with width and mass scale consistent with the Miller/Scalo estimate. The dashed curve shows the effect of adding a distribution of binary companions according to equation [5.16] with $\zeta_0 = 0.64$, $B = 1.2$, and binary fraction $\mathcal{F} = 0.75$.

6. SUMMARY AND DISCUSSION

6.1 Summary of Results

In this paper, we have presented models of the initial mass function using the idea that stars, in part, determine their own masses through the action of stellar winds and outflows. Our results can be summarized as follows:

[1] We have presented a semi-empirical mass formula (SEMF) for forming stars (see equation [2.10]). This result determines the transformation between the initial conditions for star formation and the final masses of forming stars and uses the idea that stars determine their own masses through the action of stellar winds and outflows (Figure 2; see also SLA).

[2] We have presented an empirical model for the IMF in the limit that the effective sound speed is the most important physical variable which determines stellar masses. In this limit, the spectrum of initial conditions for star formation is given by the combination of the observed clump mass distribution [3.3] and the relationships between clump mass, density, and linewidth (equations [3.1] and [3.2]). This distribution in conjunction with the SEMF produces a nearly power-law distribution of masses of forming stars (see Figure 3). This theoretical distribution is in reasonable agreement with the observed IMF.

[3] The empirical model of the IMF also allows us to estimate the overall star formation efficiency $\mathcal{E} \sim 0.07$. This calculated efficiency (see equation [3.7]) is in reasonable agreement with observations.

[4] We have studied the IMF in the limit where a large number n of physical parameters play a role in determining stellar masses. The central limit theorem shows that for any SEMF which can be written as the product of parameters (as in equation [4.1]) the resulting distribution of stellar masses (the IMF) approaches a log-normal distribution as the number of parameters $n \to \infty$. Since the observed IMF is crudely given by a log-normal distribution, this theory is in reasonable agreement with observations as long as the number n of parameters which characterize star forming environments is sufficiently large.

[5] When the central limit theorem applies (item [4]), the resulting IMF is specified by two quantities: the total width $\langle\sigma\rangle$ of the distribution and the characteristic mass scale m_C. For a given SEMF, both of these quantities can be *calculated* from the original parameters in the problem. The total width $\langle\sigma\rangle$ is determined by the quadrature sum of the variances of the distributions of all of the input parameters of the problem (see equation [4.11]). The mass scale m_C is determined by the average of the logarithms of the input parameters (see equation [4.9]). For the SEMF of §2, the values of $\langle\sigma\rangle$ and m_C estimated from observed distributions of the input parameters are in basic agreement with those required to fit the observed IMF (see Appendix B).

[6] We have studied the IMF resulting from the physically realistic case of intermediate numbers of variables ($1 < n < \infty$). In this case, the exact form for the IMF depends on the distributions of the original physical parameters of the problem. When these input parameters have uniform (flat) distributions, the convergence of the IMF to a log-normal form is quite rapid (Figures 4 and 5). For the case of power-law distributions of the initial variables, the convergence is much slower and the IMF retains a power-law tail at high masses (Figure 6).

[7] We have briefly considered the effects of binary companions on the theory of the IMF presented in this paper. We show that the inclusion of binaries does not greatly change the resulting IMF; in addition, the manner in which binaries change the IMF can be directly calculated (see §5.3 and Figure 7).

[8] The combination of all of these results demonstrates the consistency of the hypothesis that winds and outflows help determine the masses of forming stars by limiting the infall.

6.2 Discussion

At this point, we must carefully assess what we have calculated and what we have not. We have studied the idea that stars, in part, determine their own masses through the action of strong stellar winds and outflows. The question thus becomes: Have we proven this conjecture or have we presented mere speculation? The results of this paper represent much more than the latter, but unfortunately much less than the former. We have demonstrated the *plausibility* of our hypothesis, but we have certainly fallen short of a full proof. In this work, we have demonstrated that the idea of stars determining their own masses through the action of stellar outflows is in fact compatible with the observed distribution of stellar masses.

Perhaps the most important result of this work is that it provides a framework to approach the calculation of the IMF. In this framework, the calculation of the IMF involves the two steps outlined in the introduction: (1) the selection of initial conditions and (2) the transformation between a given set of initial conditions and the final mass of the star (see also Zinnecker 1989, 1990). Although treatments of both of these steps have been given here, the calculation of each of these steps can be refined considerably (see §6.5).

The semi-empirical model (§3) and the random model (§4), as presented here, are opposite limits of the same underlying problem. In the empirical limit, the distribution of sound speed *is* the distribution of initial conditions. In the opposite limit of a random model with a large number of independent variables, the total distribution of initial conditions is independent of the distribution of the sound speed except for its contribution to the overall width of the distribution. The actual physical case lies between these two extremes, i.e., the IMF should be determined by *several* variables (say, $n = 3 - 10$) which are not completely independent. However, even in the limit of a single variable – the effective sound speed – we obtain a nearly-power-law distribution which is reasonably close to the observed IMF. Furthermore, relatively few independent variables are necessary to "round out" the distribution to be even closer to the observed distribution (see Figure 5).

Next, we must consider the issue of uniqueness. In the limit that a large number of physical variables play a role in the star formation process, the central limit theorem implies that the resulting composite distribution (the IMF) always approaches a log-normal form. As a result, many different theories can, in principle, predict very nearly the same IMF. The best way to discriminate between competing theories is thus to look for the deviations of the theoretically predicted IMF from a pure log-normal form. As a general rule, the IMF will deviate most from log-normal at the *tails* of the distribution, i.e., at the low-mass and high-mass ends. It is thus crucial to obtain tight observational constraints on the IMF at both high and low masses. Unfortunately, however, the IMF is notoriously difficult to determine at both the high mass end (Massey et al. 1995) and the low mass end (Tinney 1995). This issue represents a challenge for the future.

6.3 Implications of the Theory

The picture of the IMF promoted in this paper can be tested, or at least highly constrained, by observations. In addition, this theory makes several preliminary predictions, which we discuss below.

The first issue is that of the semi-empirical mass formula. This result provides a transformation between the initial conditions for star formation and the final stellar properties. In this picture, the final stellar mass is most sensitive to the total effective sound speed and the relation has the form $M_* \sim a^\mu$ with $2 \le \mu \le 3$. This theoretical result is in good agreement with the observed correlation between the mass M_{*max} of the largest star in a region and the observed line-width (Δv) in that region (Myers & Fuller 1993); the observed relation can be written

$$M_{*max} \propto (\Delta v)^{2.38 \pm 0.17} ,$$

and is valid for the mass range $0.1 < m < 30$. The theoretical relation $M_* \sim a^\mu$ is a direct result of the hypothesis that stars help determine their masses through the action of stellar winds and outflows. Furthermore, the SEMF has the same general form for a variety of cases and hence this result is fairly robust (see Appendix A).

We have shown that as the number of variables in the SEMF increases, the form of the IMF approaches a log-normal distribution, provided only that many different variables play a role in the SEMF. The degree to which an exact log-normal distribution is obtained is illustrated by Figures 3 – 7. Thus, one prediction of this theory is that the IMF should have (nearly) a log-normal form for *any* star forming environment. Furthermore, when the IMF deviates from a pure log-normal distribution, it is expected to have a power-law tail at high masses as illustrated by Figure 6.

The mass distribution can be characterized by two parameters: the mass scale m_C and the total width $\langle\sigma\rangle$ of the distribution. For a given SEMF, the values of these parameters $\langle\sigma\rangle$ and m_C can be calculated from the distributions of the original variables in the problem (see equations [4.9] and [4.11]). Although the underlying distributions might not be known exactly, the width and central values of the distributions may be estimated. Such estimates, in conjunction with the results of this paper, may be useful in determining the IMF for models of galaxy formation, cooling flows, and other astrophysical systems.

6.4 The Low-Mass End of the IMF and Brown Dwarfs

Another important issue is the lower mass cutoff for stars. The search for brown dwarfs has interested astronomers for many decades, both as a limiting case of stellar evolution (e.g., Burrows & Liebert 1993; Laughlin & Bodenheimer 1993) and as a source of dark matter in the galactic halo (e.g., Heygi & Olive 1989; Adams & Walker 1990; Salpeter 1992; Graff & Freese 1995). In addition, microlensing experiments are now providing an important probe of these low mass stellar populations (e.g., Alcock et al. 1993; Aubourg et al. 1993). Within the paradigm of star formation invoked here, however, the formation of large numbers of brown dwarfs is difficult. In the following discussion, we examine this statement in more detail for both limits presented in §3 and §4.

In the limit that the effective sound speed is the most important physical variable (§3), the SEMF implies that in order to form very low mass stars, the effective sound speed must be very small. However, even the lowest temperatures expected in present day molecular clouds $T \sim 10$ K lead to the formation of stars with masses greater than the brown dwarf limit. Roughly speaking, stars need to become reasonably large in order to produce winds sufficiently powerful to reverse the infall; objects with masses smaller than the brown dwarf limit can only form within star forming regions with very small mass infall rates (such a scenario has been advocated by Lenzuni, Chernoff, & Salpeter 1992; see also Zinnecker 1995). We thus expect brown dwarfs to be rare.

In the opposite limit in which many different physical variables conspire to produce a nearly log-normal distribution (§4), the characteristic mass scale and total width must have given values ($m_C \approx 0.1$ and $\langle\sigma\rangle \approx 1.6$) in order to be consistent with the observed IMF. This result implies that the number of stars with masses less than m_C (and hence objects below the brown dwarf limit) is very highly suppressed. We stress that this claim

is stronger than a blind extrapolation of the observed IMF into the unknown: In the limit of large n, the distribution approaches a log-normal form and, other than m_C and $\langle\sigma\rangle$, *there are no additional parameters to specify.* For the IMF of equation [4.14], the fraction of the total mass that resides in stars with masses less than the brown dwarf limit (taken here to be $m_{BD} = 0.08$) is $\sim 5\%$. Notice also that this putative brown dwarf population corresponds to the low mass end of the usual stellar population and *not* the halo population. If brown dwarfs make up a substantial fraction of the mass of the galactic halo, then they must arise from a population with an IMF different from that of field stars.

6.5 Future Work

Many directions for future research along these lines remain. In this paper, we have presented a basic framework which can be used to calculate theoretical models of the IMF. Thus far, we have considered only extremely simple models for both the SEMF and the distribution of initial conditions. Thus, essentially all steps of the calculation can be improved significantly.

For the SEMF, more elaborate models of the type considered here can be derived (see also Appendix A). In addition, future work should use hydrodynamic simulations to study the manner in which protostellar outflows reverse the infall. Such work will help determine or constrain the form of the SEMF. Finally, one can also work backwards from the observed distributions of initial conditions and see what type of SEMF is required to produce the observed IMF.

For the distributions of initial conditions, both theoretical and observational approaches should be pursued. Almost all of the relevant physical variables appearing in the SEMF can be measured in actual star forming regions. The corresponding distributions can also be determined, e.g., the effective sound speed a (see §2), the rotation rate Ω (Goodman et al. 1993), the wind efficiency parameters $\beta/\alpha\epsilon$ (Lada 1985), and the star/disk mass fraction γ (Beckwith & Sargent 1993; Beckwith et al. 1990; Adams, Emerson, & Fuller 1990). Although some data on these distributions currently exist, much more is necessary to fully understand the problem. In addition to finding more accurate descriptions of these distributions, future observational surveys can also determine how the distributions of the variables change from one star forming environment to another.

The relative importance of the physical variables is given by the size of the contribution σ_j^2 to the total width of the distribution (see Table 1). The effective sound speed is thus the most important variable. The rotation rate Ω, the wind efficiency factors $\beta/\alpha\epsilon$, and star/disk mass fraction γ are the next most important. Future studies should prioritize their efforts accordingly, both for observational approaches (as described above), and theoretical studies, which we discuss next.

The distributions of the physical variables can also be calculated theoretically. For example, the distribution of effective sound speed can be calculated from the combination of the line-width vs density relationship (equation [3.1]) and the clump mass distribution (equation [3.3]). These distributions, in turn, can be calculated from MHD wave consid-

erations (Fatuzzo & Adams 1993; McKee & Zweibel 1995) and molecular cloud models (Norman et al. 1995). Similarly, theoretical models of the protostellar wind mechanism (Shu et al. 1994) and disk stability calculations (Laughlin 1994) will eventually determine the distributions of the dimensionless parameters α, β, γ, and ϵ appearing in the SEMF. Calculations of this type remain in their infancy; much more work must be done in order to fully understand these distributions.

In summary, we have presented a basic calculational framework which can be used to build theoretical models of the IMF. This approach is based on a SEMF and the underlying distributions of the physical variables which enter into the star formation problem. The simplest cases of these models are presented here and show reasonable agreement with the observed IMF. In the future, all steps of this calculation can be improved and we hope to eventually obtain a fundamental understanding of the IMF.

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APPENDIX A: ALTERNATE MODEL FOR THE SEMI-EMPIRICAL MASS FORMULA

In this appendix, we discuss an alternate derivation of the semi-empirical mass formula of §2. We show that the final result is very similar to that derived in the text. We thus conclude that the general form of the SEMF is fairly robust.

At a given time in the collapse, most of the newly falling material falls to radii near the centrifugal radius R_C. We therefore consider the condition that the ram pressure of the outflow is sufficiently strong to reverse the infall at the radius R_C. This condition defines the effective end of the infall phase and can be written in the form

$$\dot{M}_w v_w = \delta \dot{M} v_C \, , \tag{A1}$$

where $v_C \sim (GM/R_C)^{1/2}$ is the infall speed at the centrifugal radius and where δ is a dimensionless parameter.

We use this condition in place of equation [2.1] (the old defining equation for the end of infall) and keep the remaining assumptions concerning the scalings of outflow strengths, etc. After some rearrangement, we obtain the alternate scaling relation

$$L_* \, M_*^{1/2} \, R_*^{1/2} \, = 4 m_0 \gamma^{3/2} \delta \left(\frac{\beta}{\alpha \epsilon} \right) \frac{a^7}{G^{3/2} \Omega} = \lambda \, \frac{a^7}{G^{3/2} \Omega} \, , \tag{A2}$$

where we have defined the parameter $\lambda \sim 500$. Once again, the stellar properties appear on the left hand side of the equation and the initial conditions appear on the right hand side.

In dimensionless form, this transformation can be written

$$\widetilde{L} \, m^{1/2} \, \widetilde{R}^{1/2} \, = 2500 \lambda_3 \, a_{35}^7 \Omega_1^{-1} \, , \tag{A3}$$

where we have introduced $\widetilde{R} \equiv R_*/(1 R_\odot)$ and $\lambda_3 \equiv \lambda/10^3$.

Although this transformation appears to have a somewhat different form than that derived in §2 in the text, it leads to exactly the same scalings for the case of (low mass) young stellar objects with luminosity dominated by infall energy. In this case, the SEMF takes the form

$$m = \Lambda a_{35}^3 \Omega_1^{-2/3} \, , \tag{A4}$$

where the parameter Λ contains all of the original parameters of the problem. This form is exactly the same as that obtained in §2.

APPENDIX B: CALCULATION OF THE EXPECTED WIDTH OF A LOG-NORMAL IMF

In this Appendix, we estimate the total width $\langle\sigma\rangle$ and the mass scale m_C of the IMF based on the SEMF of §2 and observed distributions of the fundamental physical parameters.

The most important physical variable is the effective sound speed, which has the largest contribution to the total width. The distribution of the effective sound speed can be determined from the clump mass distribution [3.3] and the relation [3.2] between the effective sound speed and the clump mass. The result is

$$\frac{dN}{da} = \mathcal{N} a^{-[(p-1)q+1]} \,, \tag{B1}$$

where $\mathcal{N}$ is a normalization constant and $q \approx 4$ and $p \approx 3/2$ are the indices of the distributions [3.2] and [3.3]. Since the index appearing in the distribution [B1] is ~ 3, we need to introduce a lower cutoff a_0 to keep the distribution bounded. We take $a_0 \approx 0.20$ km/s, corresponding to the thermal sound speed at a temperature $T = 10$ K. Next, we let $x = \ln a/a_0$ and calculate the mean of the variable x:

$$\langle x\rangle = (p-1)q \int_0^\infty x\, dx\, \mathrm{e}^{-(p-1)qx} = \frac{1}{(p-1)q} \,. \tag{B2}$$

Thus, the relevant reduced variable ξ_a which shows how the sound speed enters into the IMF is given by

$$\xi_a = \mu(x - \langle x\rangle) = \mu\left\{x - \frac{1}{(p-1)q}\right\} \,, \tag{B3}$$

where μ is the exponent of the sound speed appearing in the SEMF ($\mu = 2 - 3$, depending on the mass range). Another straightforward integration then gives us the variance

$$\sigma_a^2 = \frac{\mu^2}{(p-1)^2 q^2} \approx 1.56 \,, \tag{B4}$$

where we have used $\mu = 2.5$, $p = 1.5$, and $q = 4$ to obtain the numerical estimate. Thus, the sound speed contributes a little over half of the total width of the distribution (recall that $\langle\sigma\rangle^2 = 2.45$).

Similarly, we can calculate the appropriate mean value of the sound speed from this distribution. As described in the text, we must calculate the average of the logarithm of the variable, and then exponentiate the result. We thus obtain

$$\bar{a} = \exp[\langle \ln a\rangle] = a_0 \exp[1/(q(p-1))] \approx 0.33 \text{ km s}^{-1} \,. \tag{B5}$$

Of the remaining variables, the dimensionless factors α, β, γ, etc. appearing in the SEMF can all be treated the same approximate manner. Suppose a given variable z_j varies by a factor $\mathcal{F}_j$ and enters into the SEMF with an exponent μ_j. Then, the quantity

$\xi_j = \mu_j \ln z_j$ lies in the interval $[-\mu_j \ln \mathcal{F}_j, +\mu_j \ln \mathcal{F}_j]$. The contribution to the total variance is then approximately given by

$$\sigma_j^2 = \mu_j^2 \left[\ln \mathcal{F}_j \right]^2 . \tag{B6}$$

In the following table, we list estimates for the exponents μ_j and the expected variance factors $\mathcal{F}_j$ for the variables in the SEMF. The final columns show the contribution σ_j^2 to the total width of the distribution and the mean value (as defined by equation [4.5]) which determines the characteristic mass scale. For the exponents μ_j, we use the the form [2.15a] for the SEMF. The distribution for the rotation rate Ω was derived from the results of Goodman et al. (1993). The distribution for the mechanical luminosity factors $(\beta/\alpha\epsilon)$ for protostellar outflows was taken from Figure 7 of Lada (1985); here we consider the three wind parameters α, β, and ϵ to be coupled (not independent) and hence described by a single distribution. The star/disk mass fraction γ has been estimated from various calculations of the stability of self-gravitating star/disk systems (see, e.g., Adams, Ruden, & Shu 1989; Shu et al. 1990; Laughlin 1994; Woodward, Tohline, & Hashisu 1994). Finally, the efficiency factor η for protostellar luminosities has been estimated from calculations of protostellar structure (e.g., Stahler, Shu, & Taam 1980; Palla & Stahler 1990, 1992) and from radiative transfer models of protostellar spectral energy distributions (Adams & Shu 1986; Adams, Lada, & Shu 1987; Kenyon, Calvet, & Hartmann 1993).

Using the values given in Table 1 and the SEMF, we can calculate the total width and characteristic mass scale. We find that the calculated width is $\langle \sigma \rangle \approx 1.81$, which is slightly higher than the observed value of $\langle \sigma \rangle = 1.57$. Similarly, the calculated mass scale is $m_C \approx 0.25$, which is again higher than the observed value of $m_C = 0.095$. Thus, the calculated values are approximately correct, but still differ from the observed values by a significant amount. In any case, this calculation is meant to be illustrative rather than definitive.

Table 1. Mean and Variance of Fundamental Parameters

parameter	μ_j	$\mathcal{F}_j$	σ_j^2	$\bar{\alpha}$
a_{35}	5/2	$N.A.$	1.56	0.94
Ω_1	2/3	3	0.54	1.5
$\beta/\alpha\epsilon$	1/3	10	0.60	100
γ	1	2	0.48	0.5
δ	1/3	2	0.05	1.0
η	1/3	2	0.05	0.5

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FIGURE CAPTIONS

Figure 1. Observed estimates of the initial mass function. Dashed curve shows the power-law IMF of Salpeter (1955). Solid curve shows the log-normal analytic fit to the IMF from Miller & Scalo (1979). The dashed curve with symbols shows a more recent empirical estimate of the IMF taken from Rana (1991). All three distributions are normalized to unity at $m = 1$ ($M_* = 1 M_\odot$).

Figure 2. The masses of forming stars as a function of initial conditions for the SEMF of §2. This figure shows contours of constant mass in the plane of initial conditions. The effective sound speed $a_{35} = a/(0.35 \text{ km s}^{-1})$ constitutes the horizontal axis and the rotation rate $\Omega_1 = \Omega/(1 \text{ km s}^{-1} \text{ pc}^{-1})$ constitutes the vertical axis. The remaining parameters of the problem are taken to have constant values as described in the text. The contours correspond to masses in the range $0.1 \le m \le 100$, with the mass increasing from left to right in the figure. The region in the upper left corner (above the dashed curve) corresponds to brown dwarfs; the region in the lower right corner corresponds to stars so massive that they become unstable.

Figure 3. Empirical model for the initial mass function. Dashed curve shows the IMF resulting from the semi-empirical mass formula of §2 and the observed scaling laws which describe the distribution of effective sound speed, i.e., the distribution of initial conditions. The solid curve shows the Miller/Scalo fit to the observed IMF. Both curves are normalized to unity at $m = 1$ ($M_* = 1 M_\odot$).

Figure 4. Random model for the initial mass function. Dashed curve shows the IMF resulting from the semi-empirical mass formula of §2 and a distribution of initial conditions described by a collection of $n = 10$ random variables. The solid curve shows the Miller/Scalo fit to the observed IMF. Both curves are normalized to unity at $m = 1$ ($M_* = 1 M_\odot$).

Figure 5. Random model for the IMF for different numbers n of the fundamental variables. The solid curve shows the Miller/Scalo fit to the observed IMF. Dashed curves show distributions calculated from a collection of random variables with $n = 1$, 2, 3, and 5. As the value of n increases, the curves become closer to the log-normal ($n \to \infty$) limit. All curves have been scaled so that the total width $\langle \sigma \rangle$ of the distribution and the characteristic mass scale m_C agree with the observed values from the Miller/Scalo IMF, i.e., $m_C = 0.095$ and $\langle \sigma \rangle = 1.57$. In addition, all curves are normalized to unity at $m = 1$ ($M_* = 1 M_\odot$).

Figure 6. Composite distributions for the IMF using n fundamental variables with power-law distributions. The various curves are shown for $n = 1$, 3, 10 and 20. Also shown is the log-normal curve corresponding to the limit $n \to \infty$. All curves have been scaled so that the total width $\langle \sigma \rangle$ of the distribution and the characteristic mass scale m_C agree with the observed values from the Miller/Scalo IMF, i.e., $m_C = 0.095$ and $\langle \sigma \rangle = 1.57$. In addition, all curves are normalized to unity at $m = 1$ ($M_* = 1 M_\odot$).

Figure 7. The effects of binary companions on the IMF. Solid curve shows the primary mass distribution (taken here to have the Miller/Scalo form). Dashed curve shows the effects of including an additional distribution of binary companions, where we have used the prescription of §5.3. The binary fraction $\mathcal{F} = 0.75$, the variance of the mass ratio distribution is $\sigma_{\alpha S} = 1$, and the mean of the mass ratio distribution is $\langle \ln \alpha_S \rangle = -1$. Both curves are normalized to unity at $m = 1$ ($M_* = 1 M_\odot$).